Mi penultima charla: de un milagro a otro milagro y en busca de uno mas

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CIEM , Cordoba, Marzo 17 , 2023

Para bien o para mal, ninguna parte de esta charla es el producto de CHATGPT

The signal processing motivation

In signal processing, once you specify a measuring mechanism one arrives at a GLOBAL operator, given by an integral kernel or a FULL matrix. One needs to compute many of its eigenfunctions efficiently and economically. The quality of the possible images will depend on these eigenfunctions and their eigenvalues.

In certain cases this computation can be done by a MIRACLE: one can exhibit a differential operator of second order which has the same eigenfuntions as the global operator. The numerical computation of the eigenfunctions of the differential operator, a LOCAL one, is (relatively speaking) a trivial matter compared to the initial task. It is extremely well-conditioned. The computation of the eigenfunctions of the GLOBAL operator is extremely ill-conditioned.

What is behind this miracle?

La respuesta a esta pregunta ha motivado mi trabajo por muchos agnos en colaboracion con muchos colegas arrancando con Hans Duistermaat, L. Haine y en particular con A. Tirao, I. Pacharoni, A. Duran, M. Castro, I. Zurrian, P. Iliev, M. Yakimov, R. Casper.

Before we go there.....what are these eigenfunctions/eigenvalues good for?

The spectrum of the integral operator is discrete and lies in $(0, 1)$. In most important INVERSE PROBLEMS a few eigenvalues are (say) between .2 and .9, nothing above .9. Then there is a "spectral gap" below .2 and all remaining eigenvalues are below 10^{-13}.

Any good stable reconstruction algorithm should try to find the projection of the UNKNOWN object on the linear span of the eigenfunctions with eigenvalues above the spectral gap, i.e. eigenvalues between .2 and .9.

If this span captures the important featues of the images in question we are ok. Otherwise we need to change the measuring scheme, the integral operator, etc.

Historia, prehistoria

C. Shannon, *A mathematical theory of communication*, Bell Tech. J., vol **27**, 1948, pp 379–423 (July) and pp 623–656 (Oct).

In this ground-breaking paper lying down the mathematical foundations of communication theory, Claude Shannon considers a basic problem in harmonic analysis and signal processing: how to best concentrate a function both in physical and frequency space.

You cannot have functions that are "time and band limited".

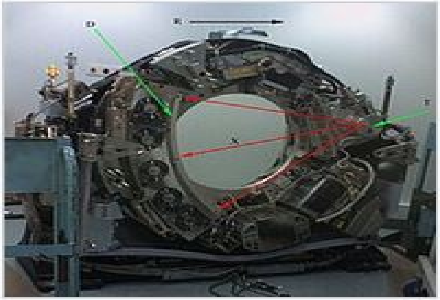
I will be talking about "signals" as in the case of C. Shannon, but these could be higher dimensional images. In fact I was led to this area by problems in medical imaging, with two or three dimensional images.

I was dealing with "limited angle tomography" where the gantry can only go partly around the patient.

CT scan

Modern CT scanner

Other names X-ray computed tomography (X-ray CT), computerized axial tomography scan (CAT scan),[1] computer aided tomography, computed tomography scan



CT scanner with cover removed to show internal components. Legend:
T: X-ray tube
D: X-ray detectors
X: X-ray beam
R: Gantry rotation

X-ray tomography

limited angle tomography

Grünbaum, F. A. *A study of Fourier space methods for "limited angle" image reconstruction*, Numer. Functional Analysis and Optimization, **2** pp 31-42, 1980.

Davison, M. E. and Grünbaum, F. A. *Tomographic reconstruction with arbitrary directions*, Comm. Pure and Applied Math. **43** pp77-120 , 1981.

NOISY data that is NOT complete.

The MAIN QUESTION

What is the relation between the **amount of data, i.e. radiation dosage** and **image quality?**

Does increasing the amount of data by 30 percent improve the quality of the image by 50 percent, or 20 percent, or only by 5 percent ?

Here is Shannon's question from scratch:

Consider an **unknown** signal $f(x)$ of finite duration, i.e. the signal is non-zero only in the interval $[-T, T]$.

The **data** you have are the values of the Fourier transform $Ff(k)$ of f for values of k in the interval $[-W, W]$.

$$(Ff)(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

In practice we only have noisy values of $Ff(k)$.

Do not even think of analytic continuation......

What is the best use you can make of this data?

More generally we want to determine f with support in A given the values of Ff in the set B. Abuse of notation

$$BFf = g = \text{known}, \qquad Af = f.$$

We can write the system of linear equation , where E is BFA,

$$Ef = BFAf = g.$$

and we need to look at the operators

$$E^*E = AF^{-1}BFA \quad \text{and} \quad EE^* = BFAF^{-1}B.$$

Gauss' NORMAL EQUATIONS

Back to Shannon :

$$(Kf)(x) = (E^*Ef)(x) = \int_{-T}^{T} \frac{\sin(W(x-y))}{x-y} f(y)\,dy, \qquad x \in A.$$

MIRACLE

The differential operator below (with appropriate boundary conditions, i.e. some selfadjoint extension)

$$(Df)(x) = ((T^2 - x^2)f'(x))' - W^2x^2f(x)$$

has simple spectrum (multiplicity one) and **COMMUTES** with the integral operator $K = E^*E$. **They have the same eigenfunctions and the differential operator has very spread out spectrum**.

Hemos visto el primer milagro

This problem was posed originally by Shannon himself but a full solution had to wait for the joint work, in different combinations of three remarkable workers at Bell labs in the 1960's: David Slepian, Henry Landau and Henry Pollak.

The differential operator above is called the "prolate spheroidal" differential operator. The Laplacian in R^3 separates in "prolate spheroidal" coordinates.

D. Slepian and H. O. Pollak, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, I*, Bell System Tech. Journal, Vol. 40, No. 1 (1961), 43–64.

H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, II*, Bell System Tech. Journal, Vol. 40, No. 1 (1961), 65–84.

H. J. Landau and H. O. Pollak, *Prolate spheroical wave functions, Fourier Analysis and Uncertainty, III*, Bell System Tech. Journal, Vol. 41, No. 4 (1962), 1295–1336.

D. Slepian, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, IV*, Bell System Tech. Journal, Vol. 43, No. 6 (1964), 3009–3058.

D. Slepian, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, V*, Bell System Tech. Journal, Vol. 57, No. 5 (1978), 1371–1430.

We saw that the operator

$$(Df)(x) = ((T^2 - x^2)f'(x))' - W^2x^2f$$

has simple and very spread out spectrum and an appropriate selfadjoint extension of D commutes with K

$$KD = DK$$

Therefore every eigenfunction of D is one of K.

This is a very useful fact: one computes NUMERICALLY the eigenfunctions of D and this can be done in a stable and efficient fashion.

Notice the role played by the function

$$\phi(x, k) = e^{ixk}$$

My question from around 1980: can one replace e^{ikx} by some other function $\phi(x, k)$ and still repeat what SLP did?

Looking for an extension of the very useful property

$$KD = DK$$

to other situations I asked the following question around 1980

If $\phi(x,k)$ are eigenfunctions of some arbitrary L

$$L\left(x, \frac{d}{dx}\right)\phi(x,k) \equiv (-D^2 + V(x))\phi(x,k) = k^2\phi(x,z)$$

and you construct a new kernel $k(x,y)$ by means of

$$k(x,y) = \int_{-W}^{W} \phi(x,k)\bar{\phi}(y,k)dk$$

when (i.e. for which V(x), besides V=0) will the resulting integral operator K acting on $L^2([-T,T])$ allow for a commuting differential operator D??

A bit of experimentation led me to see the relevance of a property of the eigenfunctions $\phi(x,k)$, and I started asking the question

Find all nontrivial instances where a function $\phi(x,k)$ satisfies

$$L\left(x,\frac{d}{dx}\right)\phi(x,k)\equiv(-D^2+V(x))\phi(x,k)=k^2\phi(x,z)$$

as well as

$$B\left(k,\frac{d}{dk}\right)\phi(x,k)\equiv\left(\sum_{i=0}^{M}b_i(k)\left(\frac{d}{dk}\right)^i\right)\phi(x,k)=\Theta(x)\phi(x,k).$$

All the functions $V(x), b_i(k), \Theta(x)$ are, in principle, arbitrary except for smoothness assumptions. Notice that here M is arbitrary (finite). **This is the BISPECTRAL PROPERTY**.

The problem in this generality was posed and solved with Hans Duistermaat in a paper that appeared in 1986. We started around 1980.

Duistermaat J. J. and Grünbaum F. A. *Differential equations in the spectral parameter*, Comm. Math. Phys. **103** (1986), 177–240.

Las tres versiones del problema biespectral

El caso racional (Bessel)

El caso periodico (Gegenbauer, Jacobi)

Tres singularidades regulares y sus casos confluentes con una singularidad irregular.

El caso doblemente periodico (Lame) La funcion P de Weierstrass.

Cuatro singulatidades regulares.

The solution of this **"bispectral problem"** has unexpected connections with many parts of mathematics.

I will mention a partial list.

SOME UNEXPECTED CONNECTIONS

Integrable systems, Korteweg deVries partial differential equations, Solitons, Toda flows, KP, Calogero Moser, etc....

Special monodromy properties of the operator $L(x, d/dx)$

The importance of the Darboux process (originating in Differential Geometry, 1884, Moutard and a big fire in the Library in Paris)

Tau functions given by characters of irreducible representations of certain groups such as $GL(N)$. Other Tau functions **still** not connected to representations of groups.

The ad-conditions, a Lie algebra nilpotency condition.

What have we seen so far?

The analysis of the COMMUTING MIRACLE behind the prolate spheroidal wave operator has led to a lot of mathematics in connection with INTEGRABLE SYSTEMS and other parts of mathematics

It is hard, but not impossible, to imagine that Slepian, Landau and Pollak predicted all these connections.

They probably did not anticipate connections with topics like the Riemann zeta function

$$\sum_{n=1}^{infinity} 1/n^z \qquad \prod_p 1/(1-p^{-z})$$

Euler, Riemann

and yet......

PNAS RESEARCH ARTICLE | MATHEMATICS

Check for updates

The UV prolate spectrum matches the zeros of zeta

Alain Connes[a,b,1] and Henri Moscovici[c]

Edited by Robion Kirby, University of California, Berkeley, CA; received December 22, 2021; accepted April 12, 2022

We describe a remarkable property of the self-adjoint extension of the prolate spheroidal operator introduced in 1998 by A.C. The restriction of this operator to the interval whose characteristic function commutes with it is well known, has a discrete positive spectrum, and is well understood. What we have discovered is that the restriction of the prolate differential operator to the complement of the finite interval admits (besides a replica of the above positive spectrum) negative eigenvalues whose ultraviolet (UV) behavior reproduces that of the squares of zeros of the Riemann zeta function. Moreover, we show that their corresponding eigenfunctions belong to the Sonin space. This feature fits with the proof (by A.C. and C. Consani) of Weil's positivity at the Archimedean place, which uses the compression of the scaling action to the Sonin space. Furthermore, we construct an isospectral family of Dirac operators whose spectra have the same UV behavior as the zeros of the Riemann zeta function.

prolate | spheroidal | Riemann | zeta

Significance

We show that the eigenvalues of the self-adjoint extension (introduced by A.C. in 1998) of the prolate spheroidal operator reproduce the UV behavior of the squares of zeros of the Riemann zeta function, and we construct an isospectral family of Dirac operators whose spectra have the same UV behavior as those zeros.

The prolate spheroidal wave functions play a key role in refs. 1–3 in relation to the Riemann zeta function. In all these applications, they appear as eigenfunctions of the angle operator between two orthogonal projections in the Hilbert space $L^2_+(\mathbb{R})$ of even square integrable function on $\mathbb{R}$. These projections depend on a parameter $\lambda > 0$; the projection P_λ is given by multiplication with the characteristic function of the interval $[-\lambda, \lambda] \subset \mathbb{R}$. The projection $\widehat{P_\lambda}$ is its conjugate by the Fourier transform $\mathbb{F}_{e_\mathbb{R}}$ which is the unitary operator in $L^2_+(\mathbb{R})$ defined by

$$\mathbb{F}_{e_\mathbb{R}}(\xi)(y) = \int \xi(x) \exp(-2\pi ixy)\, dx.$$

In all the above applications of prolate spheroidal wave functions. the miraculous existence, discovered by the Bell Labs group (4–6), of a differential operator W_λ commuting with the angle operator plays only an auxiliary role.* In the present paper, we uncover another “miracle”: a careful study of the natural self-adjoint extension of W_λ introduced in ref. 9, *Lemma 6* (see also ref. 10, section 3.3) to $L^2(\mathbb{R})$ shows that it still has a discrete spectrum and that its negative eigenvalues reproduce the ultraviolet (UV) behavior of the squares of zeros of the Riemann zeta function. In a similar way, the positive spectrum corresponds, in the UV regime, to the trivial zeros. This coincidence holds for two values $\lambda = 1$ and $\lambda = \sqrt{2}$. The conceptual reason for this coincidence is the link between the operator

$$(W_\lambda \xi)(x) = -\partial_x(\lambda^2 - x^2)\partial_x \xi(x) + (2\pi\lambda)^2 x^2 \xi(x) \qquad \textbf{[1]}$$

Youtube Dec 7 2021 Alain Connes, IHES , College de France

Prolate spheroidal functions and zeta.

Dos milagros por el precio de uno.

PNAS COMMENTARY

Serendipity strikes again

F. Alberto Grünbaum[a,1]

The breadth and depth of ref. 1 are truly remarkable.

Alain Connes has found, starting around 1998, some intriguing connections between the so-called prolate spheroidal functions and zeros of the Riemann zeta function. This is well documented in references 1–3, 9, and 10 of the paper (1) commented upon here.

The importance of the prolate functions arose in the subject of time-and-band limiting of great impact in signal communication back in the 1960s in the hands of D. Slepian, H. Landau, and H. Pollak at what was then a vibrant research center in the physical sciences, namely, Bell Labs. Central to this subject is the surprising existence of a differential operator that commutes with a naturally appearing integral operator. This is already a miracle in an area that is ripe with miracles.

The work of A. Connes, by himself and with different collaborators, has given new meaning to the word "miracle." The observation that signals that try very hard to be time limited and also band limited (as they have to be in the real world) but cannot do so for mathematical reasons would yield useful results in the study of the zeros of the zeta function of Riemann is something that should excite any mathematician, and, indeed, any scientist.

Ref. 1 adds another unexpected twist to the discoveries

Hemos visto el segundo milagro

Las tres caras del analisis de Fourier

PSF:

(1) $$\int_{-T}^{T} \frac{\sin\sigma(t-\tau)}{\pi(t-\tau)} \phi_K(\tau)\, d\tau = \lambda_K \phi_K(t), \qquad K = 0, 1, \cdots.$$

DPSS:

(2) $$\sum_{n=-M}^{M} \frac{\sin\sigma(m-n)}{\pi(m-n)} \phi_K[n] = \lambda_K \phi_K[m], \qquad K = 0, \cdots, 2M.$$

P-DPSS:

(3) $$\sum_{n=0}^{M} \frac{\sin((2K+1)(m-n)\pi/N)}{N \sin((m-n)\pi/N)} \phi_i[n] = \lambda_i \phi_i[m], \qquad i = 0, \cdots, M.$$

Bell Laboratories

600 Mountain Avenue
Murray Hill, New Jersey 07974
Phone (201) 582-3000

April 25, 1979

Professor F. Alberto Grünbaum
Mathematics Department
University of California
Berkeley, California 94720

Dear Alberto:

It was a pleasure talking with you today. I'm sure we'll be communicating again in the future.

The reprint "Estimation, etc." enclosed details how I first solved the $\frac{\sin x}{x}$ integral equation. See Appendix 2. The reprint "Group Codes, etc." might be of interest to you as it poses some nice problems about finding point configurations spread out on the sphere.

In my recent paper on discrete prolates, the eigenvalues of

$$a_{ij} = \frac{\sin 2\pi W(i-j)}{\pi(i-j)}, \quad 0 < W < \tfrac{1}{2}$$

$$i,j = 0,1,\ldots,N-1$$

are considered. In dealing with the discrete Fourier transform as you proposed here, the matrix of interest is

$$\hat{a}_{ij} = \frac{\sin 2\pi \frac{M}{N}(i-j)}{\sin 2\pi \frac{(i-j)}{N}}, \qquad M < N$$

$$i,j = 1,2,\ldots,M' < N .$$

I haven't worked at this one ever.

El caso finito: la transformada de Fourier discreta, Gauss

P un entero positivo ARBITRARIO, y la recta es reemplazada por las P raices de la unidad.

M toma el rol de W.

NN toma el rol de T.

Con esta eleccion los analogos del operador integral y el diferencial toman la forma

$r[k] = sin(((2 * M + 1) * \pi * k)/P)/sin((\pi/P) * k)$

$MM[i, j] = r[i - j]$

Esta matrix MM es el analogo del operador integral.

Definiendo

$c[i] = (-cos((\pi/P)*(2*(i-1)-2*NN)))*cos((\pi/P)*(2*M+1))$

$b[i] = sin((\pi/P) * i) * sin((\pi/P) * (2 * NN - i + 1))$

La matriz tridiagonal con c[i] en la diagonal principal y b[i] en las adyacentes conmuta con la matriz MM, y es el analogo del operador diferencial que vimos antes.

La matriz de Toeplitz que vimos con

MM[i,j]=r[i-j]

r[k] $= \sin(\alpha k)/sin(\beta k)$
que depende de **DOS** parametros libres, α, β

NO es la matriz de Toeplitz mas general que conmuta con una matrix tridiagonal con espectro simple. La mas general tiene $r[1], r[2], r[3]$ arbitrarios y para k=4,5,6,....

r[k] = r[1] $U_{k-1}(r[2]/r[1]cos(p))/U_{k-1}(cos(p))$

con U_k
los polinomios de Chebychev de segunda especie.

Esta clase de matrices dependen de **Cuatro** parametros libres.

Esto indica que hay "situaciones de conmutatividad" mas alla de situaciones biespectrales.

La matriz tridiagonal que conmuta con la matrix global es esencialmente unica.

El caso finito tiene mucho en comun con el caso de la recta, pero hay diferencias importantes.

Existen funciones que son "time and band limited".

El principo de incertidumbre de Heisenberg **no es valido**.

El milagro que falta: habra una relacion entre el espectro de la matriz tridiagonal y los ceros de la funcion zeta de Riemann para alguna curva algebraica sobre un cuerpo finito?

Gauss, Artin, Hasse, Schmidt, Weil, Deligne

La importancia de tener buen "software".

SAGE

Zeta Zeroes

February 14, 2023

[1]:
```
import matplotlib.pyplot as plt
```

[2]:
```
# work over finite field of order 37 for this example
q = 37
F = GF(q); R.<t> = F[]
```

[3]:
```
H = HyperellipticCurve(t^14 + 6*t^12 + 3*t^11 + t^5 + t^4 + 5*t^3 + t - 1)
```

[4]:
```
show(H.zeta_function())
```

[4]:

$$\frac{2565726409x^{12} + 693439570x^{11} + 163052007x^{10} + 34444040x^{9} + 6609532x^{8} + 1136344x^{7} + 217878x^{6} + 30712x^{5} +}{37x^{2} - 38x + 1}$$

[5]:
```
numerator = H.zeta_function().numerator()
print(numerator)
```

```
2565726409*x^12 + 693439570*x^11 + 163052007*x^10 + 34444040*x^9 + 6609532*x^8 +
1136344*x^7 + 217878*x^6 + 30712*x^5 + 4828*x^4 + 680*x^3 + 87*x^2 + 10*x + 1
```

[6]:
```
roots = numerator.roots(ring=ComplexField())
print(len(roots))  # how many roots does our polynomial have?
```

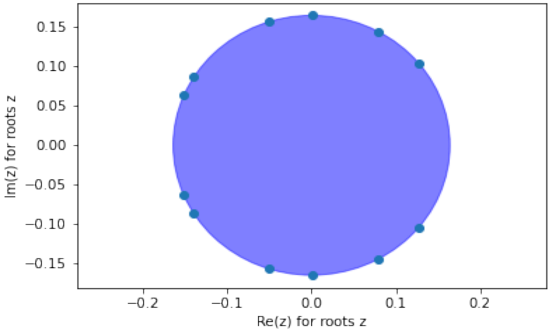
```
12
```

[7]:
```
coords = [(point[0].real(), point[0].imag()) for point in roots]
xs = [point[0] for point in coords]
ys = [point[1] for point in coords]
```

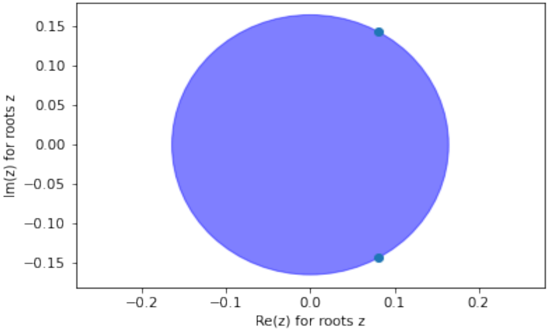
[8]:
```
# plot zeroes of the zeta function over a circle of radius 1/sqrt(q)
circle = plt.Circle((0, 0), 1/sqrt(q), color='b', alpha=0.5)
fig, ax = plt.subplots()
plt.axis('equal')
plt.xlabel("Re(z) for roots z")
plt.ylabel("Im(z) for roots z")
ax.add_patch(circle)
plt.scatter(xs, ys)
plt.show()
```

[8]:

1


0.15
0.10
0.05
0.00
−0.05
−0.10
−0.15
Im(z) for roots z
−0.2
−0.1
0.0
0.1
0.2
Re(z) for roots z

The elliptic curve $\mathbf{y}^2 = (x-1)(x-2)(x-3)(x-4), F_q with q = 37$



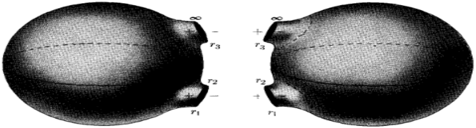


FIGURE 1–7.

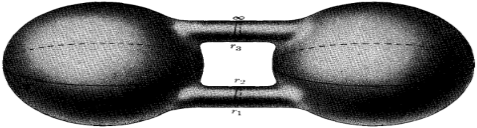
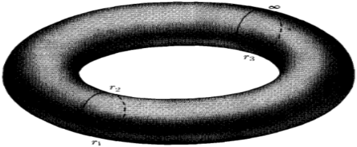


FIGURE 1–8.

1–1] ALGEBRAIC FUNCTIONS AND RIEMANN SURFACES 9



Para la curva elliptica $y^2 = (x-1)(x-2)(x-3)(x-4)$

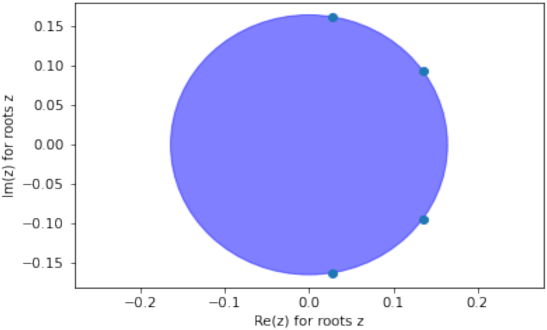
la funcion zeta de Riemann con $q = 37$ es

$$(37x^2 - 6x + 1)/(37x^2 - 38x + 1)$$

y si uno toma el logaritmo y expande en serie de Taylor, obtiene $32x + 704x^2 + 51104/3x^3 + 468864x^4 + 69335072/5x^5 + ...$

de donde se puede ver el numero de puntos racionales de la curva en distintas extensiones de F_q.

The hyperelliptic curve $\mathbf{y}^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6), F_q with q = 37$



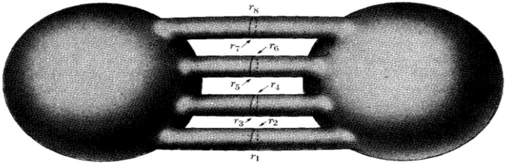


FIGURE 1–11.

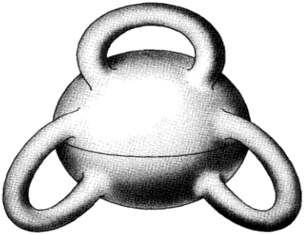
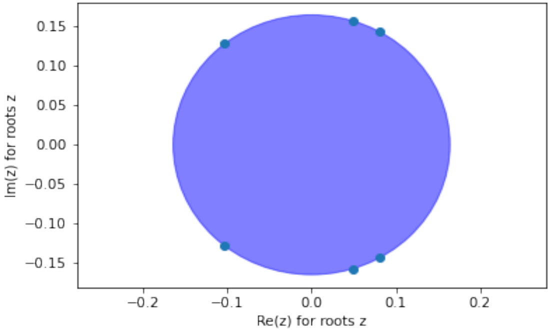


FIGURE 1–12.

The hyperelliptic curve $y^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8), F_q with q = 37$

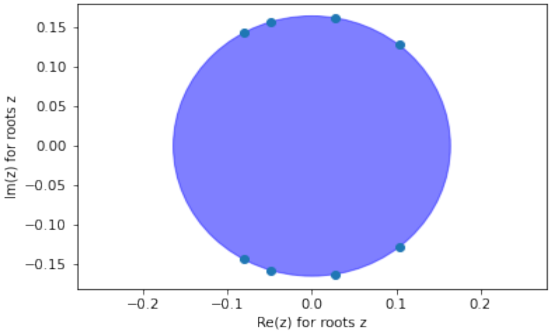


Hay muchos otros tipos de "zeta functions" no necesariamente vinculadas a curvas algebraicas (o variedades) sobre un cuerpo finito, tales como las "dynamical zeta functions" de Smale,..., Ruelle (vinculadas con sistemas dinamicos) y la funcion de Ihara y Selberg asociada a un grafico finito no orientado.

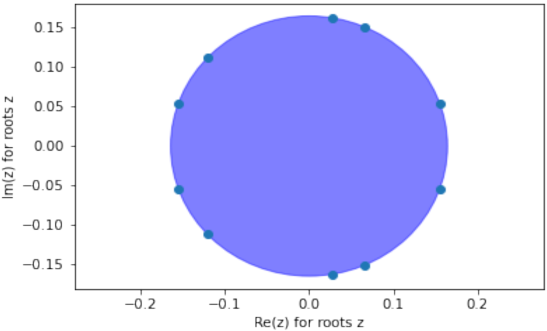
Volviendo al caso de curvas hiperelipticas sobre cuerpos finitos, calculadas con la ayuda de SAGE.
Vemos los ceros de la funcion zeta de Riemann, Artin, Weil,...

De un circulo a la recta con parte real de z igual a $1/2$ mediante un cambio de variables.

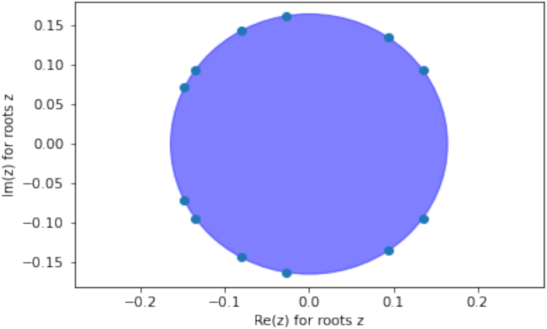
The hyperelliptic curve $\mathbf{y}^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10), F_q with q = 37$



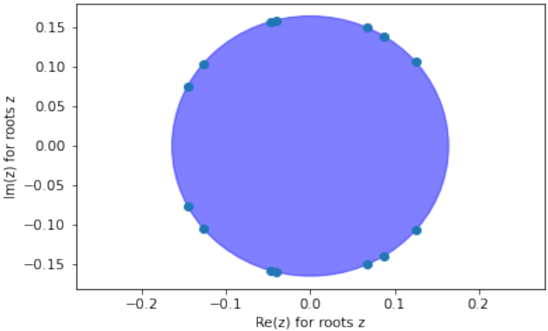
The hyperelliptic curve $\mathbf{y}^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)(x-11)(x-12), F_q with q = 37$



The hyperelliptic curve $y^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)(x-11)(x-12)(x-13)(x-14), F_q with q = 37$



The hyperelliptic curve $y^2 = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)(x-11)(x-12)(x-13)(x-14)(x-15)(x-16), F_q with q = 37$



Que hace que una funcion sea un funcion zeta?

Exponencial de una funcion generatriz de ciertos numeros naturales s_n , $n = 1, 2, 3, ...$

$$F(z) = exp(\sum_{n=1}^{infinity} s_n z^n / n)$$

tal que $F(z)$ satisfaga cierta ecuacion funcional, tenga sus ceros en lugares muy especiales cuando se extienda al plano complejo, ...

Resumen y una mirada al futuro

La busqueda de un operador "local" que comparta sus autofunciones con uno "global" y que surga de un problema natural juega un rol importante en "procesamiento de segnales" o "analisis armonico" o en **Random Matrix Theory**.

Estos **NO son** el unico lugar donde este milagro juega un rol importante.

El "Langlands program" es un esfuerzo por ver a la matematica desde muchos angulos, por ejemplo planteando cuestiones de teoria de numeros en termino de analisis armonico.

La posibilidad-necesidad de una version analitica esta expresada en "On the analytic form of the geometric theory of automorphic forms", Robert Langlands.

The geometric theory of automorphic forms was introduced by Russian mathematicians, for example, Vladimir Drinfeld, and developed by the Russian-American school,[1] but I am dissatisfied with this theory. The contemporary arithmetical theory arose in the sixties of the twentieth century from four sources: from the revival by Siegel of the investigations of the nineteenth century; from the Hecke theory; from class-field theory; from the theory of group representations, in which the names of Frobenius, Herman Weyl, and Harish-Chandra are important. In this arithmetic theory the Hecke eigenvalues are an irreplaceable element. Using these the extremely important L-functions are determined. These numbers are the eigenvalues of the Hecke operators. These operators are defined in the arithmetic theory but not in the geometric theory introduced by Gaitsgory or Frenkel.[2] The difficulty is that in the theory of the Russian-American school the eigenvectors are replaced by eigensheaves, the existence of which is difficult to establish, until now even impossible. Moreover the description of the classifying space in this theory presupposes concepts from the theory of sheaves and stacks and presupposes as well topological questions introduced in order to create a theory of classifying spaces that satisfies in large part the functorial demands of Grothendieck.

Their theory is important, but in my view it is not the theory that is necessary for expressing and proving the geometrical form of that which I call in the arithmetical theory functoriality and reciprocity. For a reason that I explain later it may be better to use the term duality, but functoriality is a consequence of duality. One of the aims of the arithmetic theory is to establish functoriality and, using this functoriality, to construct the automorphic galoisian group,[3] but in the geometric theory this group is already at hand. None the less it is necessary to show that a given group possesses the desired properties. What it is necessary to understand for the foundations of the geometric theory, introduced in this essay, is a general understanding from the sphere of sets, spaces and measures.

En recientes trabajos de P. Etingof, E. Frenkel y D. Kahzdan uno encuentra familias de operadores integrales (operadores de Hecke) que commutan con el operador diferencial de Lame.

Este es el caso eliptico, correspondiente a otro de los sistemas de coordenadas en los cuales el Laplaciano en R^3 se separa. La tesis de R. Perline, Berkeley 1984 muestra que la propiedad biespectral tradicional no se cumple en este caso.

Una nueva visita a este resultado negativo podria dar nueva vida al problema biespectral.......

Una version mas amplia de este problema, que incluya el caso eliptico, y no solo al racional y al trigonometrico podria dar resultados utiles en el programa de Langlands.

Para avanzar aca necesitamos varios milagros: con mucha suerte esto seria una escusa para que esta no sea mi penultima charla.